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1991 J. Phys. A: Math. Gen. 24 4853

(http://iopscience.iop.org/0305-4470/24/20/018)

J. Phys. A: Math. Gen. 24 (1991) 4853–4871. Printed in the UK

Spin glasses on lattices with a finite connection number

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Received 3 January 1991

Abstract. The nature of the low-temperature phase of the $\pm J$ Ising spin glass on extended Bethe lattices is studied. It is proved analytically that the distribution function of overlaps, $P(q)$, has the form $P(q) = \delta(q - \bar{q}(T, H))$ at any temperature when the open boundary condition is adopted. On the other hand, many approximate solutions are found for the equations of state for spin glasses on randomly connected lattices which are locally equivalent to the Bethe lattice. The equations of state are solved numerically. The distribution $P(q)$ in the low-temperature region is broad but still has a peak. The marginal stability of the solutions is confirmed numerically. The structure of the solution space is also investigated and consequently the present results suggest the existence of the ultra-metric structure, though the precision of data for the present restricted system size is not sufficient to exclude other possibilities.

1. Introduction

Various theoretical approaches to random spin systems are classified into essentially two categories. The first is to treat random systems in finite dimensions by numerical methods, such as Monte Carlo simulations [1,2] and the numerical transfer matrix method. The second approach is to try to find various useful concepts for spin glasses using mean-field models such as the Sherrington–Kirkpatrick (SK) model [3] and to discuss their applicability to real systems, though historically the first one follows the second. The first approach has a remarkable merit in realization of random spin systems. For example, the first confirmation [1,2] of the existence of the phase transition was performed numerically in three dimensions. The divergence of the correlation time below T_c prevents complete understanding of the nature of the low-temperature phase. On the other hand, the second approach based on studies of the SK model contributed to finding many concepts about spin glasses such as replica-symmetry breaking [4,5], ultra-metricity [6,7], etc. There are, however, several critical arguments [8,9] which suggest that the nature of the low-temperature phase of finite-dimensional systems may be quite different from that of mean-field models.

From this viewpoint, it is desirable to investigate another type of mean-field model which is expected to be more similar to real systems than the SK model. The random spin systems on the Cayley trees which we study here from this new viewpoint were originally studied by Matsubara and Sakata [10] using the method of the distribution function of an effective field. This method proved to be useful in determining various phase boundaries [10,11] in the T–p diagram with no external field. It is, however, not so powerful in determining the phase boundary in the T–H diagram. Carlson *et al* reported in [12] that all the moments of the distribution function are non-singular functions of T and H while it is shown [13] that there exists a transition line in the T–H plane on which the influence of the boundary condition reaches far inside the system. The phase boundary may correspond to the AT line for the SK model. There is, however, no such explicit theory on the present model as Parisi's theory on the SK model to clarify the nature of the low-temperature region. There is an expansion theory [14] on the present model by the replica method. This theory is expected to be correct near the critical point, but the relationship between Parisi's order parameter $q(x)$ and the distribution function of overlaps, $P(q)$, is not so straightforward in this thory as in the case of the SK model.

Recently Dewar and Mottishaw [15] studied the Cayley tree with boundary spins connected to each other using essentially the same method as Nemoto and Takayama [16] used for the SK model. Although their data have relatively large error bars, they suggested the marginal stability of solutions in a low-temperature region. Lai and Goldschmidt [17] performed Monte Carlo simulations in various networks. They suggested that for open boundary systems the distribution $P(q)$ converges to a single delta peak and they urged that the low-temperature phase is replica-symmetric. Their procedure of eliminating the boundary sum is, however, not sufficient in that they fixed the number of generations to be eliminated for all their system sizes. They also simulated the relevant systems on randomly connected networks with a finite average of connection numbers and they showed that the behaviour of $P(q)$ in the spin glass phase is qualitatively different from that in the paramagnetic phase.

In the present paper we first prove analytically that $P(q)$ for open boundary systems reduces to a single delta peak even if the careful limiting procedure is adopted. Next, for randomly connected networks with three neighbours for each spin, we show numerical results similar to, but more improved in accuracy than, those of Dewar and Mottishaw. We also discuss several features characteristic of the low-temperature phase.

2. $P(q)$ for open boundary systems

As was emphasized by one of the present authors [18, 19], the concept of emerging order induced by the increase of the system size for a fixed boundary field plays an essential role for the study of any phase transitions.

We show first the equivalence of two definitions of $P(q)$ for open boundary systems. This equivalence is proved for the SK model in [20]. Next we show that the variance of $P(q)$ vanishes at any temperature and in any homogeneous external field. The Hamiltonian of our system is

$$\mathcal{H}(\Lambda, J) \equiv - \sum_{\substack{(i,j)\\ i,j\in\tilde{\Omega}}} J_{i,j} S_i S_j - H \sum_{i\in\tilde{\Omega}} S_i - \sum_{i\in\partial\tilde{\Omega}} \Lambda_i S_i . \tag{1}$$

To define overlaps, let us consider two systems which are different from one another only by their boundary fields Λ_i ($i \in \partial\tilde{\Omega}$). As was pointed out in [12] and [15], one can expect to find bulk behaviour only in a small region far from the boundary. This can easily be seen in an example of ferromagnets on the Cayley trees [21] in which there exists no spontaneous magnetization at any temperature when the magnetization is defined over the whole system. Hence we define overlaps on a

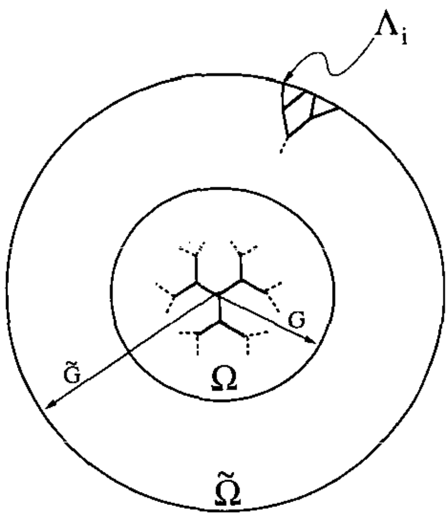

Figure 1. The support of the overlaps is a sublattice Ω which is embedded in $\tilde{\Omega}$. The random field is applied on the boundary of $\tilde{\Omega}$.

sublattice Ω with the radius G (see figure 1) and take the limit $G \to \infty$ after taking the limit $\tilde{G} \to \infty$, where $\tilde{G}$ is the radius of $\tilde{\Omega}$.

The two definitions of the distribution of overlaps are given in the following, by specifying each configuration of the boundary field by Greek letters. Namely we have

$$P(q) \equiv \left[\left\langle \delta\left(q - \frac{1}{N}\sum_{i\in\Omega} m_i^\alpha m_i^\beta\right)\right\rangle_\Lambda\right]_J \tag{2}$$

$$\tilde{P}(q) \equiv [\langle\langle \delta(q-\hat{q})\rangle\rangle_\Lambda]_J \tag{3}$$

$$\hat{q} \equiv \frac{1}{N}\sum_{i\in\Omega} S_i^\alpha S_i^\beta \tag{4}$$

in our system with the boundary fields $\{\Lambda^\alpha\}$, where $m_i^\alpha \equiv \langle S_i^\alpha\rangle$ and S_i^α is a spin variable which takes $+1$ or -1. The above expectation values are defined by

$$\langle X\rangle \equiv \frac{\mathrm{Tr}[\exp\left(-\beta\mathcal{H}(\Lambda^\alpha, J) - \beta\mathcal{H}(\Lambda^\beta, J)\right) X]}{\mathrm{Tr}[\exp\left(-\beta\mathcal{H}(\Lambda^\alpha, J) - \beta\mathcal{H}(\Lambda^\beta, J)\right)]}$$

$$\langle X\rangle_\Lambda \equiv \sum_{\{\Lambda^\alpha\},\{\Lambda^\beta\}} W\{\Lambda^\alpha\}W\{\Lambda^\beta\}X\left(\sum_{\{\Lambda^\alpha\},\{\Lambda^\beta\}} W\{\Lambda^\alpha\}W\{\Lambda^\beta\}1\right)^{-1}$$

and

$$[X]_J \equiv \sum_{\{J\}} W\{J\}X\left(\sum_{\{J\}} W\{J\}\right)^{-1}$$

where $W\{\Lambda^\alpha\}$ and $W\{J\}$ are the weights of the boundary fields and of the bond configurations, respectively. In order to show that $\tilde{P}(q) = P(q)$, it is sufficient to prove that the variance of the distribution function $P^{(\alpha,\beta)}_{\Lambda,J}(q) \equiv \langle \delta(q-\hat{q}) \rangle$ is equal to zero for any Λ^α, Λ^β and J, since the equation

$$\left\langle \delta\left(q - \frac{1}{N}\sum_i S_i^\alpha S_i^\beta\right)\right\rangle = \delta\left(q - \frac{1}{N}\sum_{i\in\Omega} m_i^\alpha m_i^\beta\right) \tag{5}$$

holds in this case. The amplitude of the thermal fluctuation of the overlap $\hat{q}$ for certain realization of the boundary-field configuration and of the bond configuration can be written by correlation functions between $\sigma_i^{\alpha\beta}$ and $\sigma_j^{\alpha\beta}$ where $\sigma_i^{\alpha\beta} \equiv S_i^\alpha S_i^\beta$. That is, we have

$$\langle \hat{q}^2 \rangle - \langle \hat{q} \rangle^2 = \frac{1}{N^2}\sum_{i,j} \langle \sigma_i^{\alpha\beta}; \sigma_j^{\alpha\beta} \rangle \tag{6}$$

$$\langle X; Y \rangle \equiv \langle XY \rangle - \langle X \rangle \langle Y \rangle . \tag{7}$$

As shown in appendix A, the quantity $|\langle \sigma_i^{\alpha\beta}; \sigma_j^{\alpha\beta} \rangle|$ is bounded from above by $[\tanh(\beta J_{\max})]^{R_{ij}}$ with the maximum absolute value of the coupling, $J_{\max}$. Thus it is easily shown that $\langle \hat{q}^2 \rangle - \langle \hat{q} \rangle^2$ converges to zero in the thermodynamic limit. Hence the equivalence of the two definitions is confirmed.

Next, we show how $P(q)$ in (3) reduces to a single delta function. Only the outline of the proof on this reduction is shown here and details are described in appendices. Let us first present the definitions of the relevant quantities, as follows:

$$\begin{aligned} q^{\alpha\beta}(\Lambda, J) &\equiv \frac{1}{N}\sum_{k\in\Omega} \langle S_k^\alpha \rangle \langle S_k^\beta \rangle \\ q_j^{\alpha\beta}(S_{[j]}^\alpha, S_{[j]}^\beta, \Lambda_j, J_j) &\equiv \frac{1}{N(g)}\sum_{k\in\Omega_j} \langle S_k^\alpha \rangle_{\mathcal{H}_j} \langle S_k^\beta \rangle_{\mathcal{H}_j} \end{aligned} \tag{8}$$

and

$$q_j^{\alpha\beta}(\Lambda_j, J_j) \equiv \sum_{S_{[j]}^\alpha, S_{[j]}^\beta} w(S_{[j]}^\alpha) w(S_{[j]}^\beta) q_j^{\alpha\beta}(S_{[j]}^\alpha, S_{[j]}^\beta, \Lambda_j, J_j). \tag{9}$$

Here, $[j]$ indicates the 'mother' site of the site j, and Ω_j (or $\tilde{\Omega}_j$) is a subset of Ω (or $\tilde{\Omega}$) which are connected to the centre through the site j (see figure 2). The symbol Λ_j in such expressions as $q_j^{\alpha\beta}(\Lambda_j, J_j)$ indicates all the surface-field variables charged on $\partial\tilde{\Omega}_j$, and, similarly, J_j indicates all the bonds that have at least one end point in $\tilde{\Omega}_j$. Although these symbols are the same as local variables at the site labelled by 'j', there is no fear of confusion since the meanings of the symbol will be clear from the contexts. The weight $w(S_{[j]}^\alpha)$ is an arbitrary distribution function, and $\langle \cdots \rangle_{\mathcal{H}_j}$ is defined by

$$\begin{aligned} \langle X \rangle_{\mathcal{H}_j} &\equiv \frac{\mathrm{Tr}_{\tilde{\Omega}_j}[\exp(-\beta\mathcal{H}_j(\Lambda_j^\alpha, J_j) - \beta\mathcal{H}_j(\Lambda_j^\beta, J_j))X]}{\mathrm{Tr}_{\tilde{\Omega}_j}[\exp(-\beta\mathcal{H}_j(\Lambda_j^\alpha, J_j) - \beta\mathcal{H}_j(\Lambda_j^\beta, J_j))]} \\ \mathcal{H}_j(\Lambda_j, J_j) &\equiv -\sum_{k\in\tilde{\Omega}_j} J_{k,[k]} S_k S_{[k]} - \sum_{k\in\tilde{\Omega}_j} H S_k - \sum_{k\in\partial\tilde{\Omega}_j} \Lambda_k S_k . \end{aligned} \tag{10}$$

In addition, let us define several moments of the distribution functions.

$$\begin{aligned} &v \equiv l - m^2 \qquad l \equiv \left[\left\langle q(\Lambda, J)^2\right\rangle_\Lambda\right]_J \qquad m \equiv \left[\langle q(\Lambda, J)\rangle_\Lambda\right]_J \\ &v^{(g)} \equiv l^{(g)} - (m^{(g)})^2 \\ &l^{(g)} \equiv \left[\left\langle q_j(\Lambda_j, J_j)^2\right\rangle_{\Lambda_j}\right]_{J_j} \qquad m^{(g)} \equiv \left[\left\langle q_j(\Lambda_j, J_j)\right\rangle_{\Lambda_j}\right]_{J_j} \end{aligned} \tag{11}$$

where j is an arbitrary site located $(g-1)$ steps inside from the boundary $\partial\Omega$. Here we have omitted the replica indices α and β.

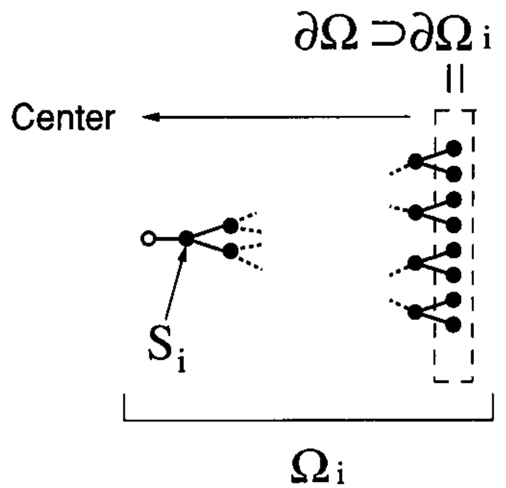

Figure 2. Ω_i is a branch which is connected to the site i. The overlap $q_i(\Lambda_i, J_i)$ in the text is defined on Ω_i.

Then we can show the following equation.

$$v = \frac{1}{z} v^{(G)} + \mathrm{O}((z-1)^{-G}, t_{\max}^{G}). \tag{12}$$

Similarly, the variance $v^{(g)}$ can be evaluated using the following recursive equation.

$$v^{(g+1)} = \frac{1}{z-1} v^{(g)} + \mathrm{O}((z-1)^{-g}, t_{\max}^{g}) \qquad (g = 1, 2, \ldots, G-1) \tag{13}$$

It can be seen by solving these two equations that the variance v converges to zero in the limit $G \to \infty$. The details are given in appendix 2.

It should be noted that the absence of variance which has been proved here is essentially different from the fact that there exists no finite spontaneous magnetization in a ferromagnetic spin system on the Cayley tree [21]. The reason for no spontaneous magnetization in the latter case is that the 'magnetization' was defined over the whole system $\tilde{\Omega}$. If we define magnetization only on Ω similarly to the above definition of the overlaps, we find that a finite magnetization is induced by an infinitesimal magnetic field applied on $\partial\tilde{\Omega}$.

As we have mentioned in section 1, there exists a certain phase boundary in the T–H plane corresponding to the AT line. The low-temperature region is characterized by the fact that an infinitesimal fluctuation of random fields on the boundary $\partial\tilde{\Omega}$ induces a finite fluctuation in the expectation value of each spin in Ω. On the other hand, it has been clarified that the distribution function $P(q)$ for the open boundary systems reduces to a single delta function at any temperature and in any homogeneous external field. In this sense, we may call the low-temperature phase of open boundary systems 'replica-symmetric'. We can also conclude from these facts that there exist many phases in the low-temperature region and that the expectation values of spins change independently, in essence, when one jumps from one phase to another. As a result, the relative fluctuation of the macroscopic overlap function becomes vanishing below the critical temperature.

3. Randomly connected systems and numerical calculations

In this section we discuss spin glasses with 'closed' boundary conditions. There are several versions of this type of boundary conditions. For example, one can define Cayley trees with connected boundaries [14] in which the boundary sites are connected randomly to each other so that the connection number may be equal to z for each site. In these lattices there is no topological equivalence among all the sites and the lattice has one special 'centre' site (or bond). Another possible choice is to consider randomly connected networks [16] with a finite average of connection number in which each arbitrary pair of sites are connected with probability z/N, where N is the number of sites. One can also define randomly connected networks with a finite fixed connection number in which $zN/2$ bonds are distributed randomly with the constraint that z bonds should meet at each site. We believe that the essential properties derived from the last two definitions are qualitatively the same in a large enough system. In a finite system, however, some peculiar lattices which make the result obscure are more likely to be generated by the second definition than by the last one, because of a finite ratio of sites at which less than (more than) z bonds meet. In the present paper we adopt the last definition.

We investigate, by means of numerical calculations, the $\pm J$ Ising model on the networks with $z = 3$ defined above. The distribution of the coupling is taken to be symmetric (i.e. $P(J = \pm 1) = \frac{1}{2}$). Unfortunately there is no satisfactory argument on the relationship between the solutions of the equations of state for finite systems and the pure states in the thermodynamic limit. Here we assume that two types of configurations of finite systems correspond to the configurations of pure states in the limit. Therefore we search the configurations which satisfy one of the following two conditions.

$$\text{(I)} \qquad \frac{\partial f}{\partial m_i} = 0 \tag{14}$$

$$\text{(II)} \qquad \sum_j \frac{\partial^2 f}{\partial m_i \partial m_j} \frac{\partial f}{\partial m_j} = 0 . \tag{15}$$

We call hereafter the configurations satisfing the conditions I and II, of type I and of type II, respectively. It is natural to expect that the configurations of type I which minimize locally the free energy are continuously connected to the pure states in the thermodynamic limit. On the other hand, the configurations of type II minimize the free energy with the constraint that the susceptibility matrix should be singular.

Therefore, it is also natural and is supported numerically [15] for the SK model that the configurations of type II correspond to pure states if the pure states are marginally stable as is conjectured [22]. This marginal stability for the present system is supported by the calculations for the solutions of type I, as will be presented below. These above-mentioned strategies are originally adopted in the investigations of the SK model by Nemoto and Takayama [16]. The conditions (14) and (15) can be stated more simply. Namely, the solutions should give the local minima of the squared norm $g\{m\}$ of the gradient of the free energy defined by

$$g\{m\} \equiv \sum_i \left(\frac{\partial}{\partial m_i} f\{m\} \right)^2 . \tag{16}$$

The free energy and its derivatives are given as follows [23]:

$$\begin{aligned}\beta f\{m\} \equiv & \sum_i \left(\frac{1+m_i}{2} \log \frac{1+m_i}{2} + \frac{1-m_i}{2} \log \frac{1-m_i}{2} \right) \\ & + \sum_{(ij)} \left\{ \log \left[\frac{1}{1+\xi_{ij}\xi_{ji}/t_{ij}} \left(\frac{(1-t_{ij}^2)(1-\xi_{ij}^2/t_{ij}^2)(1-\xi_{ji}^2/t_{ji}^2)}{(1-m_i^2)(1-m_j^2)} \right)^{1/2} \right] \right. \\ & \left. + \left[\tanh^{-1} \frac{\xi_{ji}}{t_{ji}} - \tanh^{-1} m_i \right] m_i + \left[\tanh^{-1} \frac{\xi_{ij}}{t_{ij}} - \tanh^{-1} m_j \right] m_j \right\} \\ & - \beta H \sum_i m_i \end{aligned} \tag{17}$$

$$\beta f_i \equiv \beta \frac{\partial f}{\partial m_i} = \tanh^{-1} m_i - \beta H - \sum_j \tanh^{-1} \xi_{ij} \tag{18}$$

$$\beta f_{ii} \equiv \beta \frac{\partial^2 f}{\partial {m_i}^2} = \frac{1}{1-m_i^2} + \sum_j \frac{1}{\left[(1-t_{ij}^2)^2 - 4t_{ij}\mu_{ij}\mu_{ji}\right]^{1/2}} \frac{t_{ij}^2 - \xi_{ij}^2}{1-\xi_{ij}^2} \tag{19}$$

$$\beta f_{ij} \equiv \beta \frac{\partial^2 f}{\partial m_i \partial m_j} = \frac{-t_{ij}}{\left[(1-t_{ij}^2)^2 - 4t_{ij}\mu_{ij}\mu_{ji}\right]^{1/2}} \tag{20}$$

where

$$\xi_{ij} \equiv \frac{(1-t_{ij}^2) - \left[(1-t_{ij}^2)^2 - 4t_{ij}\mu_{ij}\mu_{ji}\right]^{1/2}}{2\mu_{ij}}$$

$$\mu_{ij} \equiv m_i - t_{ij} m_j \qquad t_{ij} \equiv \tanh \beta J_{ij}$$

We can obtain the equation of state by setting the right-hand side of [18] equal to zero. It should be noted that we have assumed here, even in the low-temperature region, the applicability of the equation of state (i.e. [18]) which is obtained by supposing that there is no multi-body effective field. That is, we have assumed that tracing out the degrees of freedom of all the spins except one site and its nearest-neighbours results in the appearance of certain effective fields on the nearest neighbour sites. It is clear that this assumption is not correct in finite systems, because there exist many loops with finite length which induce multi-body effective fields. In the thermodynamic limit, however, the probability is zero that we find at least one finite loop which includes a certain spin. Thus we can expect that there exists a certain condition under which the above assumption is correct in the thermodynamic limit. For example, this assumption probably holds in the paramagnetic phase in which there is essentially only one solution. We believe also that there is a validity condition on the equation of state which corresponds to the condition for the SK model given in [22]. This validity condition is equivalent to the convergence condition of the expansion series of Gibbs' free energy with respect to the coupling. The boundary of the convergence region is located at the point where the susceptibility matrix is singular [24]. Thus we eliminate solutions for which the Hessian matrix (f_{ij}) has negative eigenvalues, because these solutions are expected to lie beyond the validity condition.

In our explicit calculations we adopted the Marquardt method for minimizing the function $g\{m_i\}$ using its first and second derivatives. We have also used the steepest-descent method when the Marquardt method is unstable.

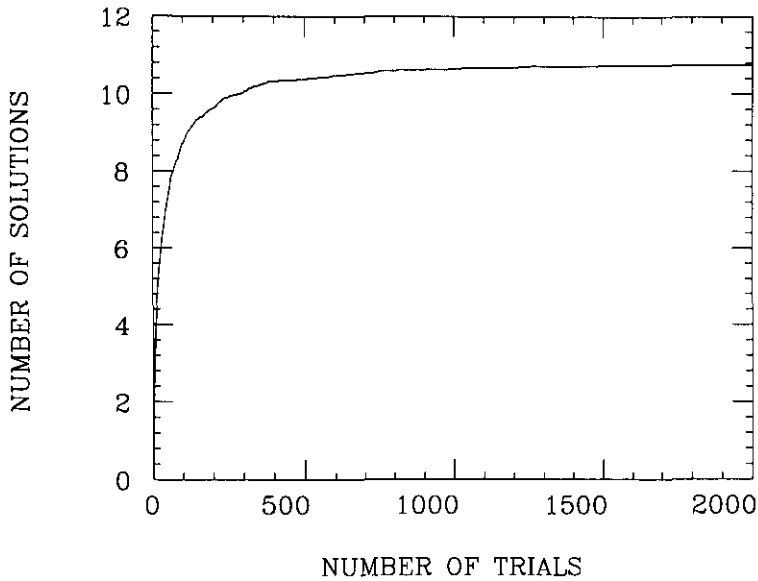

Figure 3. Number of solutions ($\bar{N}_s$) versus number of trials. $N = 20$ and $T = 0.5$.

We have investigated finite systems with the numbers of spins, 12, 16, 20, 24, 28 and 40. For each system size we generated 100 bond configurations. For each of these configurations, 100 through 4000 trials were performed. One trial starts with generating the initial spin configuration $\{m_i\}$, next the m_i's are changed one by one so that the exchange energy may be lowered until the stable configuration is reached, and then all the m_i are changed simultaneously by the above algorithm to lower the function $g\{m_i\}$. During the first two of these three processes, the $\{m_i\}$ are restricted to be $+1$ or -1. The numbers of trials are determined so that almost all the solutions may be found. The dependence of the numbers of solutions on the typical time (numbers of trials performed) is shown in figure 3 for $N = 20$ and $T = 0.5$. Since we could not afford a sufficiently large number of trials to find almost all the solutions for $N = 40$, we did not use the data for $N = 40$ in estimating the coefficient $\tilde{\alpha}$ defined below. The numbers $N_s^{\rm I}$ of solutions of type I are found to be

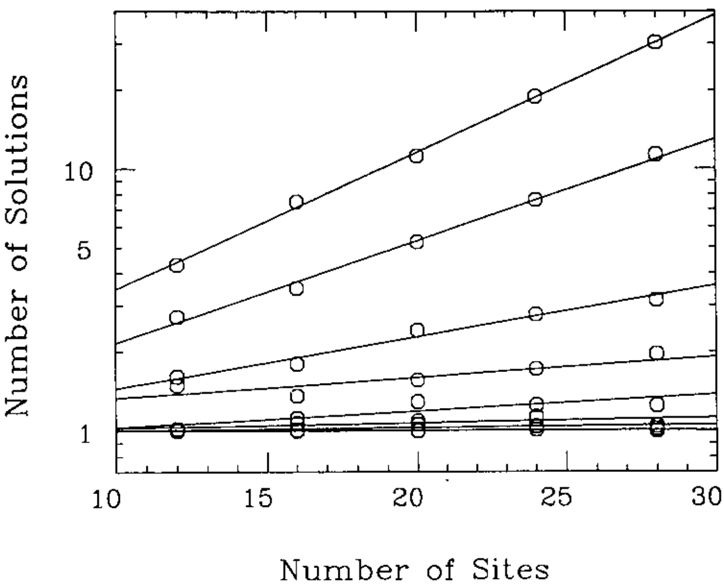

Figure 4. Total number of solutions $\tilde{N}_s = N_s^{\mathrm{I}} + N_s^{\mathrm{II}}$ versus system size. $T = 0.5$, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1 and 1.2 from the top to the bottom.

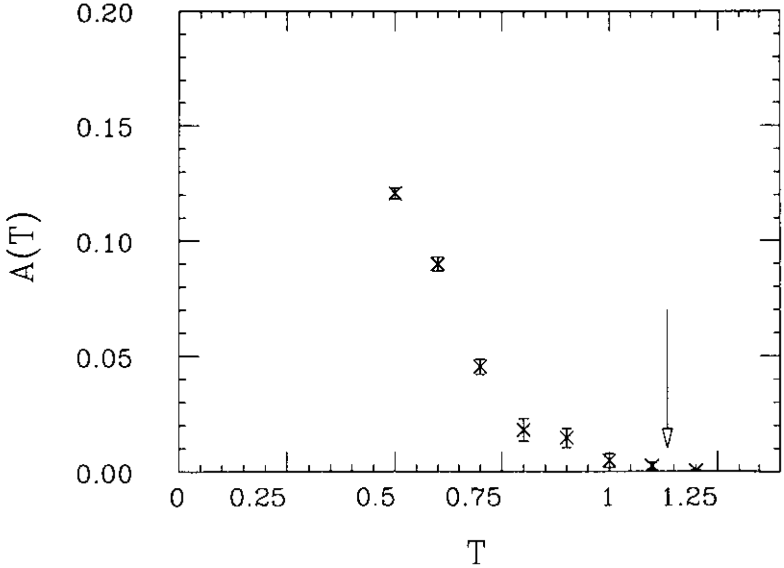

Figure 5. The coefficient $\tilde{\alpha}(T)$ defined by $\tilde{N}_s(T, N) \propto \exp(\tilde{\alpha}(T)N)$ versus temperature. The exact critical point T_c is $1/\tanh^{-1}(1/\sqrt{2}) \approx 1.13$.

very small compared to those of solutions of type II (N_s^{II}). There are only $0, 1$ or 2 solutions of this type for each bond configuration at any temperature below T_c. The size dependence of $\tilde{N}_s \equiv N_s^{\mathrm{I}} + N_s^{\mathrm{II}}$ is shown in figure 4 for various temperatures. The data are well fitted by the exponential law as is also the case with those for the solutions of the TAP equation (25).

In figure 5 we show the coefficient $\tilde{\alpha}(T)$ defined by

$$\tilde{N}_s(T,N) \propto \exp(\tilde{\alpha}(T)N). \tag{21}$$

Although it must be noted that we cannot identfy $\tilde{N}_s$ with the true number of pure states N_s because of the degeneracy of solutions in the limit $N \to \infty$, we can see from figure 5 that the onset of non-zero $\tilde{\alpha}(T)$ occurs at the critical point T_c.

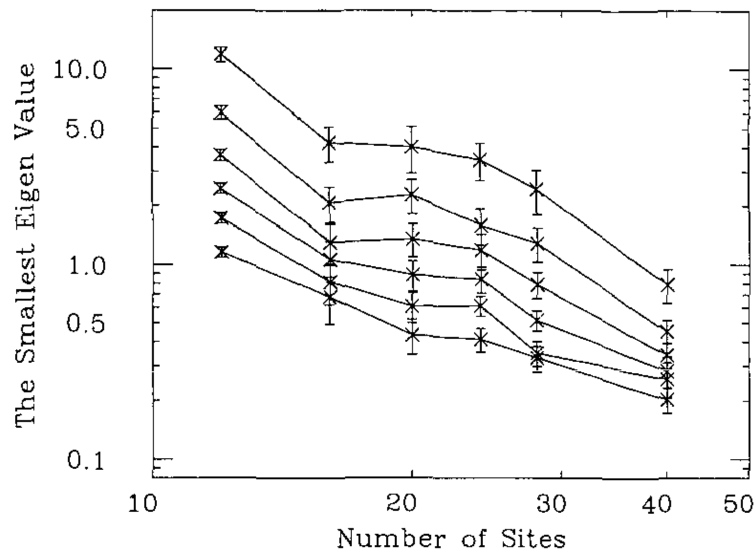

Figure 6. The smallest eigenvalues of the Hessian matrix f_{ij} for type I solutions. $T = 0.5, 0.6, 0.7, 0.8, 0.9$ and 1.0 from the top to the bottom.

In figure 6 we show the average of the smallest eigenvalues for the solutions of type I. We can see from this figure that the smallest eigenvalues tend to zero when the system size becomes large. This supports the conjecture of the marginal stability mentioned above.

We also calculate the distribution function of the overlaps

$$P(q) \equiv \left[\sum_{\mu,\nu} w_\mu w_\nu \delta\left(q - \frac{1}{N}\sum_i m_i^\mu m_i^\nu\right)\right]_J \tag{22}$$

where

$$w_\mu \equiv \exp(-\beta F\{m_i^\mu\}) \left(\sum_\nu \exp(-\beta F\{m_i^\mu\})\right)^{-1}. \tag{23}$$

Here μ and ν specify an arbitrary pair of solutions obtained. In figure 7, $P(q)$ is shown for $N = 40$ and $T = 0.6$. As is expected, the overlaps in the low-temperature region have broad distributions, correspondingly to the existence of many pure states in the thermodynamic limit. The size dependence of $P(q)$ is not clear enough to predict in detail the shape of this function in the thermodynamic limit. More

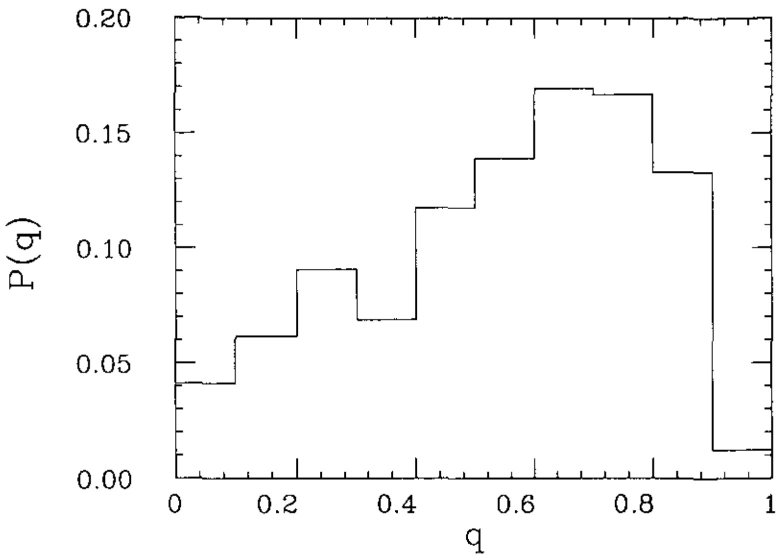

Figure 7. The distribution function of overlaps for $N = 40$ and $T = 0.6$.

calculations for larger systems are required in order to determine $P(q)$ precisely. On the other hand, above the critical temperature, $P(q)$ is a single delta function, because in this region there is only one solution ($m_i = 0$ for all i) for any bond configuration.

Furthermore we examine the ultra-metric structure which is proposed for the structure of the solution space of the SK model. We show in figure 8 the following distribution function.

$$P(q_1, q_2) \equiv \int \mathrm{d}q_3(P(q_1, q_2, q_3) + P(q_2, q_1, q_3))/2 \tag{24}$$

where

$$\begin{aligned} P(q_1, q_2, q_3) \equiv \sum_{\mu,\nu,\rho} & w_\mu w_\nu w_\rho \delta(q_1 - \min(\{q^{\mu\nu}, q^{\nu\rho}, q^{\rho\mu}\})) \\ & \times \delta(q_2 - \mathrm{mid}(\{q^{\mu\nu}, q^{\nu\rho}, q^{\rho\mu}\})) \\ & \times \delta(q_3 - \max(\{q^{\mu\nu}, q^{\nu\rho}, q^{\rho\mu}\})). \end{aligned} \tag{25}$$

We can see, from figure 8, that there exists a strong positive correlation between q_1 and q_2 suggesting the ultra-metricity

$$q_1 = q_2 \geqslant q_3 \tag{26}$$

though we can not exclude other possibilities.

4. Summary and discussions

We have proved that the distribution function of overlaps $P(q)$ consists of a single delta peak in the case of the Cayley trees with random external fields applied on their

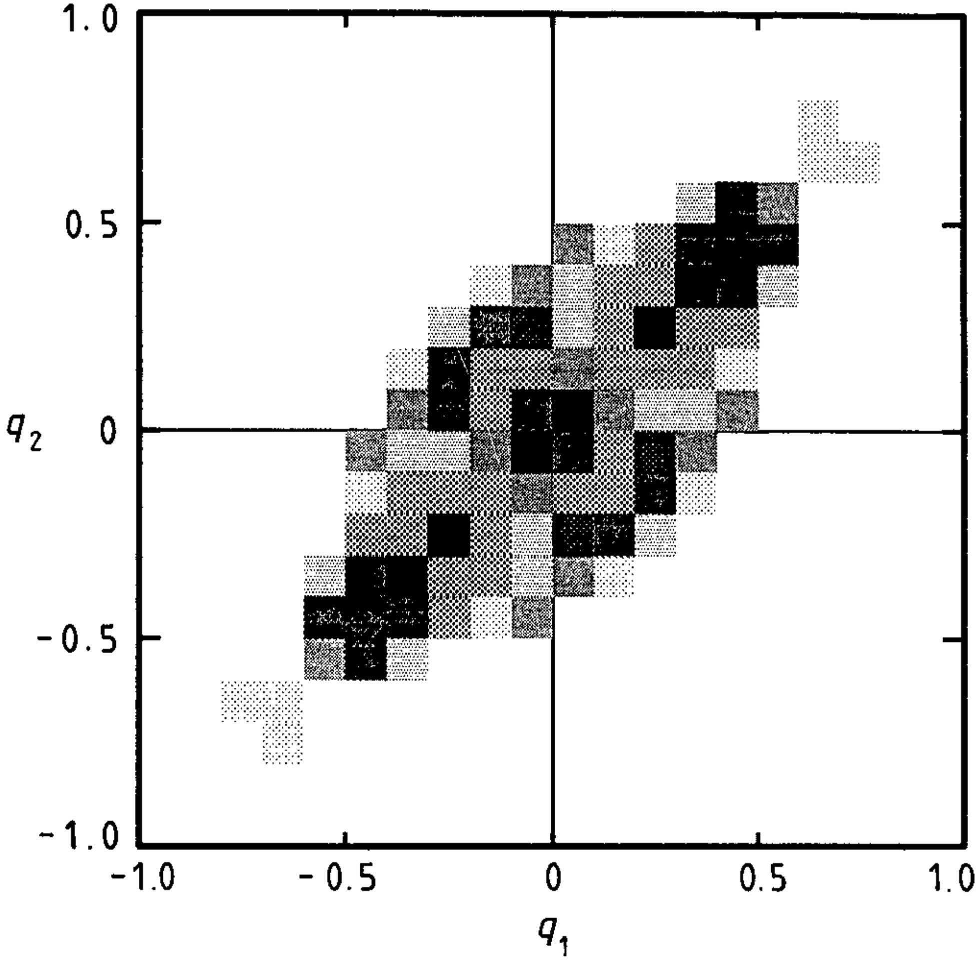

Figure 8. The distribution function $P(q_1, q_2)$ defined in the text. The degree of darkness of the boxes indicates the magnitude of $P(q_1, q_2)$. For example, white space means that $P(q_1, q_2) < 0.25$ and the darkest boxes mean that $P(q_1, q_2) > 2.00$. Here we have studied the case that $T = 0.5$ and $N = 40$.

open boundaries. On the other hand, $P(q)$ has a non-trivial structure in the closed boundary case. We can interpret these facts as follows: Even in the closed boundary case the local structure of the lattice is the same as in the open boundary case as far as the system size is infinite; that is, for an arbitrary spin one can find a Cayley tree embedded in the system with an arbitrary size which contains the spin as its centre. In such cases, every pair of boundary spins of this Cayley tree is connected by very long chains of bonds outside the tree. These chains are responsible for the different behaviours caused by the different boundary conditions. In the paramagnetic phase, however, the range of the correlation is short and these long chains have no influence on the spins at their end-points. As a result, there is no difference between the open and the closed boundaries in the paramagnetic phase. At the critical point, the range of the correlation becomes infinite. At this point, the correlation caused by the long chains outside becomes finite and, at the same time, the influence of the boundary fields reaches the centre spin. Hence the nature of the open boundary system differs from that of the closed boundary system in the low-temperature region.

In both two cases there are many phases below T_c. In the open boundary case, however, the expectation values of spins change independently to each other when one jumps from one phase to another. As a result, $P(q)$ reduces to a single delta peak. It is well known that in higher dimensions than three the ferromagnetic Ising model has many phases characterized by domain walls although we do not call it replica-symmetry breaking. We consider that the many phases which have been observed by charging infinitesimal random fields are similar to these domain wall phases. From this point of view, we should not call the low-temperature phase of the open boundary systems replica-symmetry breaking.

On the other hand, the expectation values change coherently in the closed boundary case and $P(q)$ has a non-trivial structure. The marginal stability of the solutions has also been confirmed numerically in this case (section 3). The logarithm of the number of solutions is proportional to the system size and the coefficient $\tilde{\alpha}$ is finite below T_c. It is also suggested that the space of solutions has an ultra-metric structure.

It will be interesting to determine the AT line by the present procedure which has been calculated so far only near the critical point T_c [10, 13] and at $T = 0$ [26]. More calculations for larger systems are required for more precise investigations of the low-temperature region, for example, on the determination of the asymptotic form of $P(q)$ in the thermodynamic limit, and on the further confirmation of ultra-metric structures. Concerning the number of pure states, we have only been able to show in the present paper the number of local minima of the squared norm of the gradient, which may be overcounting. We may take the number N_s^I of local minima of the free energy as the number of pure states in the limit $N \to \infty$. We also calculated the configurations which give local minima of the energy, though we have not presented them here, and we found that the numbers N_s^I for various values of the system size are fitted by the exponential law as well as $\tilde{N}_s$ mentioned in section 3. The coefficient α is found to be approximately 0.21 at $T = 0$. Thus if N_s^I gives a correct estimation of N_s and if N_s^I is not singular at $T = 0$ we can conclude that the number N_s for the present system obeys the exponential law as in the case of the SK model, though the validity of these assumptions is questionable. In any case at finite temperatures especially near the critical point, the number N_s^I that we have found here is too small to discuss the asymptotic behaviour. There even remains a possibility of non-exponential increase of N_s as a function of N. Thus the determination of the total number of solutions of the present system with larger size is also an interesting problem in future.

Finally, it is worth mentioning the relationship between the present models and the finite-dimensional models. The closed boundary system is very similar to the SK model as we have seen above. There have been reported, however, several results which suggest that the nature of the low-temperature phase of the finite-dimensional systems is quite different from that of the SK model. For example, the numerical calculations by Ogielski [1] and Bhatt and Young [2] suggest that even below T_c the EA order-parameter tends to zero in the thermodynamic limit in three dimensions. This fact shows a clear discrepancy from the prediction on the SK model. The open boundary system is closer to the finite-dimensional systems at least in this point, although we can conclude no more about the relationship between them at the present stage.

Acknowledgments

The authors are grateful to Professor H Takayama and Dr K Nemoto for useful comments and discussions. One of the authors (NK) has benefitted from suggestions by Dr M Katori and thanks also Dr N Ito for stimulating discussions.

Appendix 1. The thermal fluctuation of the overlaps

In order to estimate the upper bound of the correlation function $\langle \sigma_i^{\alpha\beta}; \sigma_j^{\alpha\beta} \rangle$ let us rewrite this quantity as follows:

$$\langle \sigma_i^{\alpha\beta}; \sigma_j^{\alpha\beta} \rangle = \Gamma_{ij}^{\alpha}\Gamma_{ij}^{\beta} + m_i^{\alpha} m_j^{\alpha} \Gamma_{ij}^{\beta} + m_i^{\beta} m_j^{\beta} \Gamma_{ij}^{\alpha} \tag{27}$$

where $\Gamma_{ij}^{\alpha} \equiv \langle S_i^{\alpha}; S_j^{\alpha} \rangle$. Thus we have

$$|\langle \sigma_i^{\alpha\beta}; \sigma_j^{\alpha\beta} \rangle| < 3 \max(|\Gamma_{ij}^{\alpha}|, |\Gamma_{ij}^{\beta}|) \tag{28}$$

because $|\Gamma_{ij}^{\alpha}| < 1, |\Gamma_{ij}^{\beta}| < 1$, and $|m_i| < 1$.

Let us estimate the correlation $\Gamma_{ij}^{\alpha} \equiv \langle S_i^{\alpha}; S_j^{\alpha} \rangle$. It is obvious that there is one and only one chain of bonds which connects the site i and the site j. Let us refer to the sites along this chain as $i_1, i_2, \ldots$ and i_R where $i_1 \equiv i$ and $i_R \equiv j$. Then we can trace out all the degrees of freedom except for the spins on this chain. Thus, we obtain a set of effective fields on these spins. That is, for our present purpose, it is enough to consider the following chain-Hamiltonian:

$$\mathcal{H}_{\text{chain}} \equiv -\sum_{p=1}^{R-1} J_{i_p, i_{p+1}} S_{i_p} S_{i_{p+1}} - \sum_{p=1}^{R} (H + H_{i_p}) S_{i_p}. \tag{29}$$

Here, H_{i_p} is the above-mentioned effective field. Next, let us trace out S_{i_p} $(p = 2, 3, 4, \ldots, R-1)$. The following simple formula is useful for this elimination.

Let K_{ml}, K_{ln} and K_{mn} be real numbers related to each other by

$$\exp(-K_{mn} S_m S_n) \equiv \text{constant} \times \operatorname*{Tr}_{S_l} \exp(-K_{ml} S_m S_l - K_{ln} S_l S_n - \eta_l S_l) \tag{30}$$

where η_l is an arbitrary real number. Then we have

$$|\tanh(K_{ij})| \leqslant |\tanh(K_{ik})||\tanh(K_{kj})|. \tag{31}$$

Using this formula repeatedly we can get the upper bound for $|\tilde{J}_{ij}|$. Here $\tilde{J}_{ij}$ is the effective coupling between the sites i and j which results from the elimination of S_{i_p} $(p = 2, 3, 4, \ldots, R-1)$. That is, we get the following pair-Hamiltonian.

$$\tilde{\mathcal{H}}_{ij} \equiv -\tilde{J}_{ij} S_i S_j - (H + H_i) S_i - (H + H_j) S_j \tag{32}$$

with

$$|\tanh \beta \tilde{J}_{ij}| \leqslant \prod_{i=1}^{R-1} |\tanh \beta J_{i_p, i_{p+1}}|. \tag{33}$$

Using this Hamiltonian, we can estimate the correlation function Γ^{α}_{ij}:

$$
\begin{aligned}
|\Gamma^{\alpha}_{ij}| &= |\langle S_i S_j\rangle_{\tilde{\mathcal{H}}_{ij}} - \langle S_i\rangle_{\tilde{\mathcal{H}}_{ij}} \langle S_j\rangle_{\tilde{\mathcal{H}}_{ij}}| \\
&= \left|\frac{\tilde{t}_{ij}(1-\eta_i^2)(1-\eta_j^2)}{(1+\tilde{t}_{ij}\eta_i\eta_j)^2}\right| \\
&\leqslant |\tilde{t}_{ij}|
\end{aligned}
\tag{34}
$$

where

$$
\tilde{t}_{ij} \equiv \tanh(\beta\tilde{J}_{ij}) \qquad \eta_i \equiv \tanh(\beta(H+H_i)). \tag{35}
$$

At last, we arrive at

$$
|\Gamma^{\alpha}_{ij}| \leqslant t_{\max}^{R_{ij}}. \tag{36}
$$

This leads to the inequality

$$
|\langle \sigma_i ; \sigma_j\rangle| < 3 \times (t_{\max})^{R_{ij}}. \tag{37}
$$

Thus we have

$$
\begin{aligned}
\langle \hat{q}^2\rangle - \langle \hat{q}\rangle^2 &= \frac{1}{N^2}\sum_{\substack{i,j\\ i,j\in\Omega}} \left\langle \sigma_i^{\alpha\beta}; \sigma_j^{\alpha\beta}\right\rangle \leqslant \frac{3}{N^2}\sum_{\substack{i,j\\ i,j\in\Omega}} t_{\max}^{R_{ij}} \\
&\leqslant \frac{3}{N}\sum_{i\in\Omega} t_{\max}^{R_{0i}} = \frac{3}{N}\left(1 + z t_{\max}\frac{1-(z-1)^G t_{\max}^G}{1-(z-1)t_{\max}}\right) \\
&\to 0 \qquad (G\to\infty).
\end{aligned}
\tag{38}
$$

In the above derivation, we have assumed that there is an upper bound for the absolute values of J_{ij}. This does not hold for the Gaussian distribution. We can, however, modify the distribution by limiting the range of distribution in a finite interval even in such cases. We believe that if we take this interval sufficiently wide such a modification does not change the essencial nature of the system.

Appendix 2. The derivation of the recursive inequality of the variances

First, let us prove the following lemma which is a key tool for the proof of (12) and (13).

Lemma. Suppose an Ising spin system is composed of several parts connected to each other by a single site. Let us indicate this site by the letter c and each part by Ω_a $(a = 1, 2, \ldots, z)$. The Hamiltonian of the whole system is

$$
\begin{aligned}
\mathcal{H} &= \mathcal{H}_c(S_c) + \sum_a \mathcal{H}_a \\
\mathcal{H}_a &= -\sum_{(i,j);i,j\in\bar{\Omega}_a} J_{ij} S_i S_j - H\sum_{i\in\Omega_a} S_i
\end{aligned}
\tag{39}
$$

where $\bar{\Omega}_a \equiv \Omega_a \cup \{c\}$. Then the following inequality holds:

$$\langle A \rangle_{\mathcal{H}} \approx \langle A \rangle_{\mathcal{H}_a} \pm \Delta A \qquad (40)$$

$$\langle \cdots \rangle_{\mathcal{H}_a} \equiv \frac{\mathrm{Tr}\, \mathrm{e}^{-\beta \mathcal{H}_a} \ldots}{\mathrm{Tr}\, \mathrm{e}^{-\beta \mathcal{H}_a}}$$

for an arbitrary value of S_c in the first term of the right-hand side. Here, $x \approx y \pm z$ stands for $y - |z| \leqslant x \leqslant y + |z|$, A is an arbitrary quantity which is defined on Ω_a and

$$\Delta A \equiv \max_{S_c} \langle A \rangle_{\mathcal{H}_a} - \min_{S_c} \langle A \rangle_{\mathcal{H}_a} . \qquad (41)$$

This lemma can be proved as follows.

$$\begin{aligned} \langle A \rangle_{\mathcal{H}} &= \frac{\mathrm{Tr}_{S_c} \mathrm{e}^{-\beta \mathcal{H}_c} \left(\prod_{p \neq a} \mathrm{Tr}_{\Omega_p} \mathrm{e}^{-\beta \mathcal{H}_p} \right) \mathrm{Tr}_{\Omega_a} \mathrm{e}^{-\beta \mathcal{H}_a} A}{\mathrm{Tr}_{S_c} \mathrm{e}^{-\beta \mathcal{H}_c} \left(\prod_{p \neq a} \mathrm{Tr}_{\Omega_p} \mathrm{e}^{-\beta \mathcal{H}_p} \right) \mathrm{Tr}_{\Omega_a} \mathrm{e}^{-\beta \mathcal{H}_a}} \\ &= \mathop{\mathrm{Tr}}_{S_c} w(S_c) \langle A \rangle_{\mathcal{H}_a} \end{aligned} \qquad (42)$$

where

$$w(S_c) \equiv \frac{\mathrm{e}^{-\beta \mathcal{H}_c} \left(\prod_p \mathrm{Tr}_{\Omega_p} \mathrm{e}^{-\beta \mathcal{H}_p} \right)}{\mathrm{Tr}_{S_c} \mathrm{e}^{-\beta \mathcal{H}_c} \left(\prod_p \mathrm{Tr}_{\Omega_p} \mathrm{e}^{-\beta \mathcal{H}_p} \right)} . \qquad (43)$$

Equation (42) leads to

$$\min_{S_c} \langle A \rangle_{\mathcal{H}_a} \leqslant \langle A \rangle_{\mathcal{H}} \leqslant \max_{S_c} \langle A \rangle_{\mathcal{H}_a} . \qquad (44)$$

It can easily be seen that (40) is a weaker statement than this.

Next, let the lattice in the above lemma be a Cayley tree, c be a certain site labelled by $[j]$, Ω_a be the set of spins which was denoted as Ω_j in section 2, and A be a certain spin S_i ($i \in \Omega_j$). In this case, we can estimate the error as follows:

$$\begin{aligned} \Delta m_i &\equiv \max_{S_{[j]}} \langle S_i \rangle_{\mathcal{H}_j} - \min_{S_{[j]}} \langle S_i \rangle_{\mathcal{H}_j} \\ &= \max_{S_{[j]}} \langle S_i \rangle_{\tilde{\mathcal{H}}_{[j]i}} - \min_{S_{[j]}} \langle S_i \rangle_{\tilde{\mathcal{H}}_{[j]i}} \end{aligned} \qquad (45)$$

with the pair-Hamiltonian $\tilde{\mathcal{H}}_{[j]i}$ defined by (32). Similarly to the derivation of (34), we get

$$\langle S_i \rangle_{\tilde{\mathcal{H}}_{[j]i}} = \frac{\eta_i + \tilde{t}_{[j]i} S_{[j]}}{1 + \eta_i \tilde{t}_{[j]i} S_{[j]}} . \qquad (46)$$

Therefore, we find

$$\Delta m_i = 2 |\tilde{t}_{[j]i}| \left| \frac{1 - \eta_i^2}{1 - \tilde{t}_{[j]i}^2 \eta_i^2} \right| \leqslant 2 |\tilde{t}_{[j]i}| \leqslant 2 t_{\max}^{R_{i[j]}} \qquad (47)$$

using (45). Thus we have

$$\langle S_i \rangle_{\mathcal{H}} \approx \langle S_i \rangle_{\mathcal{H}_j} \pm 2 t_{\max}^{R_{i[j]}} . \tag{48}$$

Note that the estimated error does not depend on the boundary field or the bond configuration.

Now, we are at the position to estimate the average and the variance of the overlaps. Let us first consider the average value of the overlaps.

$$\begin{aligned} m &\equiv [\langle q(\Lambda, J) \rangle_\Lambda]_J \\ q(\Lambda, J) &\equiv \frac{1}{N} \sum_i m_i(\Lambda^\alpha, J) m_i(\Lambda^\beta, J) \\ &= \frac{1}{N} m_0(\Lambda^\alpha, J) m_0(\Lambda^\beta, J) + \sum_{j;[j]=0} \sum_{k \in \Omega_j} m_k(\Lambda^\alpha, J) m_k(\Lambda^\beta, J) \end{aligned} \tag{49}$$

where

$$m_k(\Lambda^\alpha, J) \equiv \langle S_k \rangle_{\mathcal{H}} . \tag{50}$$

Using inequality (48), we get

$$\begin{aligned} m_k(\Lambda^\alpha, J) &\approx m_k(\Lambda_j^\alpha, J_j) \pm 2 t_{\max}^{R_{0k}} \\ m_k(\Lambda_j^\alpha, J_j) &\equiv \langle S_k \rangle_{\mathcal{H}_j} \end{aligned} \tag{51}$$

if $[j] = 0$. Strictly speaking, the quantity $m_k(\Lambda_j^\alpha, J_j)$ depends on $S_{[j]}^\alpha$. We have omitted it because the following discussion holds for any value of $S_{[j]}^\alpha$. One can also take $m_k(\Lambda_j^\alpha, J_j)$ as the value averaged over $S_{[j]}^\alpha$ with an arbitrary non-negative weight.

Then (49) becomes

$$\begin{aligned} q(\Lambda, J) &\approx \frac{1}{N} \left\{ \pm 1 + \sum_{j;[j]=0} \sum_{k \in \Omega_j} [m_k(\Lambda_j^\alpha, J_j) m_k(\Lambda_j^\beta, J_j) \pm 6 t_{\max}^{R_{0k}}] \right\} \\ &\approx \frac{1}{N} \sum_{j;[j]=0} \sum_{k \in \Omega_j} m_k(\Lambda_j^\alpha, J_j) m_k(\Lambda_j^\beta, J_j) \pm 6\chi \\ &\approx \frac{N^{(G)}}{N} \sum_{j;[j]=0} q_j(\Lambda_j, J_j) \pm 6\chi \end{aligned} \tag{52}$$

where

$$\begin{aligned} \chi &\equiv \frac{1}{N} \sum_{i \in \Omega} t_{\max}^{R_{0i}} \\ &= \frac{1 + z t_{\max} (b^G t_{\max}^G - 1)/(b t_{\max} - 1)}{1 + z(b^G - 1)/(b - 1)} \qquad (b \equiv z - 1). \end{aligned} \tag{53}$$

Note that χ does not depend on Λ or J and that it converges to zero when G goes to the infinity. Thus, by averaging (52) over Λ and J, we get

$$m \approx \frac{N^{(G)}}{N} z m^{(G)} \pm 6\chi. \tag{54}$$

As for the variance v, we have

$$\begin{aligned} v &= \left[\langle q(\Lambda, J)^2\rangle_\Lambda\right]_J - m^2 \\ &\approx \left[\left\langle (\frac{N^{(G)}}{N}\sum_j q_j(\Lambda_j, J_j) \pm 6\chi)^2 \right\rangle_\Lambda\right]_J - m^2 \\ &\approx \left[\left\langle (\frac{N^{(G)}}{N}\sum_j q_j(\Lambda_j, J_j))^2 \right\rangle_\Lambda\right]_J - m^2 \pm 12\chi \pm 36\chi^2 \\ &\approx \left(\frac{N^{(G)}}{N}\right)^2 \{z l^{(G)} + z(z-1)(m^{(G)})^2\} - m^2 \pm 48\chi \\ &\approx \left(\frac{N^{(G)}}{N}\right)^2 z v^{(G)} \pm 96\chi. \end{aligned} \tag{55}$$

Here we have used $\chi \leqslant 1$ and $(N^{(G)}/N)\sum_j q_j(\Lambda_j, J_j) \leqslant 1$. We have also used the inequality (54) to derive the last line. From the above inequalty, we can see that v vanishes in the limit $G \to \infty$ if $v^{(G)} \to 0$ in this limit because $N^{(G)}/N$ converges to $1/z$.

Similarly, $v^{(g)}$ can be expressed by $v^{(g-1)}$ as follows:

$$v^{(g)} \approx \left(\frac{N^{(g-1)}}{N^{(g)}}\right)^2 (z-1) v^{(g-1)} \pm 96\chi^{(g)}. \tag{56}$$

Now, the error term is proportional to

$$\chi^{(g)} \equiv \frac{1}{N^{(g)}} \sum_{i \in \Omega_j} t_{\max}^{R_{ij}} = \frac{(b-1)}{(b^g-1)} \frac{(bt_{\max})^g - 1}{(bt_{\max} - 1)} \qquad (b \equiv z-1) \tag{57}$$

where j is an arbitrary site located $(g-1)$ steps inside from $\partial\Omega$. It can easily be seen that there exists such a finite real constant A that

$$96\chi^{(g)} \leqslant A r^g \qquad \text{(for any } g) \tag{58}$$

when

$$r > \max(b^{-1}, t_{\max}) \tag{59}$$

Finally, we arrive at

$$v^{(g)} < (1/b) v^{(g-1)} + A r^g \tag{60}$$

since $N^{(g-1)}/N^{(g)} < 1/b$. Let r be less than unity. (This choice is possible as far as $t_{\max} < 1$.) Then it is obvious from this inequality that

$$v^{(g)} \to 0 \qquad (g \to \infty). \tag{61}$$

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